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THE EFFECT OF FINITE CONDUCTIVITY ON THE PROPAGATION OF HYDROMAGNETIC SLOW WAVES

Tosiya Taniuti

September 15, 1961

AEC Research and Development Report

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Abstract

The present paper deals with the one-dimensional propagation of disturbances in an inviscid conducting fluid of finite magnetic Reynolds number. The following conspicuous property is shown: The basic equations are not hyperbolic but nevertheless the slow wave has a domain of dependence determined by the sound velocity.

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The Effect of Finite Conductivity on the Propagation of Hydromagnetic Slow Waves

1. Introduction

The effect of finite conductivity on the propagation of hydromagnetic waves of small amplitudes [1] has been investigated by Cole [2] and Whitham [3] for the transverse case in which the magnetic field is perpendicular to the direction of wave propagation. The present paper deals with one dimensional propagation in a general configuration where the magnetic field is oblique to the direction of wave motion. Even if the fluid is inviscid, the finite conductivity makes the basic equations parabolic so that in the usual sense the wave front proceeds with infinite speed and the existence of the domain of dependence will be lost completely. In the present case this is true for the fast wave; the slow wave, however, has a domain of dependence determined by the sound speed, the wave front of which proceeds with finite speed. In Section 2 the dispersion relation is derived, in Sections 3 and 4 the above mentioned peculiar property of the slow wave is shown on the basis of the Laplace transform, and in the last section the effect of the Hall current is taken into account.

The author would like to express his gratitude to Dr. C. S. Gardner for his advice.

2. The Basic Equations and the Assumptions.

In this paper we consider the one dimensional propagation of disturbances in a quiescent conducting fluid in a constant external magnetic field H_o so that all the quantities will be assumed to be the functions of x and t only. The effect of viscosity will be neglected throughout, while that of finite conductivity will be taken into account; however, in small disturbances the joule heating resulting from the finite conductivity can also be neglected as higher order, hence we may assume that the flow is isentropic.

The coordinate system is so introduced that the z-component of H_o, H_{zo}, is zero. The disturbances of the fluid velocity, the density, and the magnetic field are denoted by v_x, v_y, v_z, ρ, and H_y, H_z respectively, while the unperturbed constant density and the corresponding sound velocity are denoted by ρ_o and a_o respectively. By virtue of the divergenceless condition of the magnetic field, the disturbance H_x can be put equal to zero. Neglecting the terms quadratic with respect to the disturbed quantities we have the linearized system of equations[*]

$$\frac{\partial \rho}{\partial t} = -\rho_o \frac{\partial v_x}{\partial x} \tag{2.1}$$

$$\rho_o \frac{\partial v_x}{\partial t} = - a_o^2 \frac{\partial \rho}{\partial x} - \frac{\mu}{4\pi} H_{yo} \frac{\partial H_y}{\partial x} \tag{2.2}$$

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[*] In this paper we employ Gaussian units and μ denotes the magnetic permeability, c the velocity of light in the vacuum.

$$(2.3)\qquad \rho_o \frac{\partial v_y}{\partial t} = \frac{\mu}{4\pi} H_{xo} \frac{\partial H_y}{\partial x}$$

$$(2.4)\qquad \frac{\partial H_y}{\partial t} = H_{xo} \frac{\partial v_y}{\partial x} - H_{yo} \frac{\partial v_x}{\partial x} + \frac{c^2}{4\pi\mu\sigma} \frac{\partial^2 H_y}{\partial x^2}$$

$$(2.5a)\qquad \rho_o \frac{\partial v_z}{\partial t} = \frac{\mu}{4\pi} H_{xo} \frac{\partial H_z}{\partial x}$$

$$(2.5b)\qquad \frac{\partial H_z}{\partial t} = H_{xo} \frac{\partial v_z}{\partial x} + \frac{c^2}{4\pi\mu\sigma} \frac{\partial^2 H_z}{\partial x^2}$$

in which the ordinary Ohmic law for the electric field vector E and the current j,

$$(2.6)\qquad \sigma(E + \frac{\mu}{c}[v \times H]) = j$$

is assumed. Introducing the dimensionless quantities denoted by a prime through the equations

$$x = Lx', \quad t = Tt', \quad H_y = H_{yo}H_y', \quad \rho = \rho_o\rho',$$

$$a_o = LT^{-1}, \quad \sqrt{\frac{\mu H_{xo}^2}{4\pi\rho_o}} \equiv b_{xo} = b_{xo}' \, LT^{-1},$$

$$\sqrt{\frac{\mu H_{yo}^2}{4\pi\rho_o}} = b_{yo} = b_{yo}' \, LT^{-1}, \qquad \frac{\mu|H_o|}{\sqrt{4\pi\rho_o}} = b_o = b_o' \, LT^{-1},$$

and eliminating v_x and v_y from Eqs. (2.1) - (2.4), we get

$$(2.7a)\qquad \left(\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2}\right)\rho' - b_{yo}'^2 \frac{\partial^2 H_y}{\partial x'^2} = 0$$

$$(2.7b)\qquad \left(\frac{\partial^2}{\partial t'^2} - b_o'^2 \frac{\partial^2}{\partial x'^2}\right) H_y' - \frac{\partial^2 \rho'}{\partial x'^2} - R_m^{-1} \frac{\partial^2 H_y'}{\partial x'^2 \partial t'} = 0$$

in which Rm stands for the magnetic Reynolds number, given by $Rm = \frac{a_o L\, 4\pi\mu\sigma}{c^2}$.

In what follows all the primes of these dimensionless quantities will be omitted unless otherwise stated. The system of equations (2.7 a,b) shows the coupling between the density compression and the magnetic disturbance, and in the limit as $Rm \rightarrow \infty$, leads to the fast and slow waves [1], while equations (2.5) represent the transverse wave [1] and are still separated from the remaining equations even for finite Rm.

In order to obtain the dispersion relation resulting from equation (2.7) we put

$$\begin{pmatrix} \rho \\ H_y \end{pmatrix} \sim e^{pt+qx} \qquad .$$

Inserting the above expression into equations (2.7 a,b) and denoting b_{xo}^2, b_{yo}^2 and b_o^2 by r_x, r_y and r respectively, we obtain the secular equation

$$(2.8)\qquad q^2 = \frac{p^2}{2} (r_x + Rm^{-1}p)^{-1}\Big[1 + r + Rm^{-1}p \pm\sqrt{(1 + r + Rm^{-1}p)^2 - 4(r_x + Rm^{-1}p)}\ \Big] \quad .$$

We now introduce the complex phase velocities λ_1 and λ_2 through the equations

$$(2.9)\qquad \lambda_{1,2}^{-1} = \Big[\frac{1}{2} (r_x + Rm^{-1}p)^{-1}\Big\{1 + r + Rm^{-1}p \mp \sqrt{(1 + r + Rm^{-1}p)^2 - 4(r_x + Rm^{-1}p)}\Big\}\Big]^{1/2}$$

in which subscripts 1 and 2 correspond to the minus and the plus signs in the right hand side respectively; it is easy to see that λ_1 and λ_2 tend to the fast and the slow speed [1] as Rm approaches infinity. Then for any complex value of p, q is given by the equation,

$$(2.8)' \qquad q_i = \pm \frac{p}{\lambda_i(p)} , \qquad (i = i,2) .$$

3. A Boundary and Initial Value Problem

We now consider an initial value and boundary value problem, defined as follows. The boundary condition: at $x = 0$, U is given as a function of time in the following way,

$$\begin{array}{ll} (A.1) & \\ & U(o,t) \begin{cases} = U_o(t) & \text{for } t > 0 \\ = 0 & \text{for } t < 0 \, ; \end{cases} \\ (A.2) & \end{array}$$

the initial condition: at $t = 0$, U and U_t are zero for positive x, namely,

$$(B) \qquad \left.\begin{array}{l} U(x,o) = 0 \\ U_t(x,o) = 0 \end{array}\right\} \text{for } x > 0$$

where U is a column vector defined by

$$(3.1) \qquad U = \begin{pmatrix} \rho \\ H_y \end{pmatrix} .$$

The solution for the positive value of x, satisfying the above conditions (A) and (B) can be given by the Laplace transform,

$$U(x,t) \sim \frac{1}{2\pi i} \int_L e^{pt+qx}\, U(p)dp \tag{3.2}$$

in which q is given by q_i of equation (2.8). (The complete solution is, of course, the superposition of the solutions corresponding to the various q_i's.) The contour L is the Bromwich path, with $\mathrm{Re}\, p > 0$, to the right of all singularities of the integrand.* Deforming the integration path to encircle the positive half plane, one can see immediately that the boundary condition (A.2) is automatically satisfied, while the initial condition (B) requires that $\mathrm{Re}\, q$ must be negative for $\mathrm{Re}\, p \longrightarrow \infty$. Since for $|p| \longrightarrow \infty$, λ_i's behave as

$$\lambda_1^{-1} \longrightarrow (Rm^{-1}\, p)^{-1\;2} \tag{3.3a}$$

$$\lambda_2^{-1} \longrightarrow 1 \tag{3.3b}$$

the above requirement admits among the q_i's of equation (2.8) those with negative sign,

$$q_i = -\frac{p}{\lambda_i}, \qquad (i = 1,2)\,. \tag{3.4}$$

The solution is the superposition of the elementary solutions with q_i's given by equation (3.4) which clearly expresses the outgoing wave. From equation (2.7a) it follows that the solution takes the form,

---

* These singularities comprise not only those of $U(p)$ but also the branch points of q which are given by
$Rm^{-1}p = -r_x$, $\qquad Rm^{-1}p = 1 - r \pm 2i\sqrt{r_y}$.

$$U(x,t) = \frac{1}{2\pi i}\int_L S(p,x)S_o^{-1}\,U_o(p)e^{pt}\,dp \tag{3.5}$$

in which $S(p,x)$ is defined by

$$S(p,x) = \begin{pmatrix} \gamma_y\, e^{q_1 x} & \gamma_y\, e^{q_2 x} \\ (\lambda_1^2 - 1)e^{q_1 x} & (\lambda_2^2 - 1)e^{q_2 x} \end{pmatrix}, \tag{3.6a}$$

S_o^{-1} is equal to $S^{-1}(p,o)$ and has the representation

$$S_o^{-1} = \begin{pmatrix} -\dfrac{\lambda_2^2 - 1}{\lambda_1^2 - \lambda_2^2}\Big/\gamma_y & \dfrac{1}{\lambda_1^2 - \lambda_2^2} \\ \dfrac{\lambda_1^2 - 1}{\lambda_1^2 - \lambda_2^2}\Big/\gamma_y & -\dfrac{1}{\lambda_1^2 - \lambda_2^2} \end{pmatrix} \tag{3.6b}$$

while $U_o(p)$ is the Laplace transform of $U_o(t)$,

$$U_o(p) = \int_o^\infty e^{-pt}\,U_o(t)dt. \tag{3.6c}$$

Equation (3.5) can also be written as

$$U(x,t) = \frac{1}{2\pi i}\int_L (A_1 e^{pt+q_1 x} + A_2\, e^{pt+q_2 x})dp \tag{3.7}$$

if the column vectors A_1 and A_2 are introduced through the equations

(3.8a) $$A_1 = (S_o^{-1}\, U_o(p))_1 \begin{pmatrix} \gamma_y \\ \\ \lambda_1^2 - 1 \end{pmatrix}$$

(3.8b) $$A_2 = (S_o^{-1}\, U_o(p))_2 \begin{pmatrix} \gamma_y \\ \\ \lambda_2^2 - 1 \end{pmatrix}$$

where $(S_o^{-1}\, U_o(p))_i$, $(i = 1, 2)$ denotes the i^{th} component of the column vector $S_o^{-1}\, U_o(p)$.

4. The Domain of Dependence of the Slow Wave

Let us first evaluate the integral for the slow mode characterized by e^{pt+q_2x} in equation (3.7).

From equation (3.3b) it follows that q_2 tends to $-p$, consequently $pt+q_2x$, to $p(t-x)$, as $\text{Re}\ p$ approaches $+\infty$. Hence deforming the path L to encircle the positive half plane, one can see easily that the integral $\frac{1}{2\pi i}\int_L A_2 e^{pt + q_2x}\, dp$ vanishes if $t-x < 0$. This shows the striking feature that even for the finite value of conductivity the slow disturbance has a domain of dependence and does not propagate with a velocity larger than the sonic one. On the other hand, the fast mode behaves in an usual sense; namely, from equation (3.3a) q_1 behaves as $-\sqrt{p}$ as $\text{Re}\ p$ tends to infinity and the same deformation of path does not lead to the existence of a domain of dependence.

It seems worth while to check these conclusions by the method of steepest descent, for the asymptotic behavior of large t. In what follows we consider the case that the external magnetic field is neither parallel nor normal to the x axis so that both r_x and r_y differ appreciably from zero. The special configuration $r = 1$ will also be excluded from our consideration. Let us for a while assume that $Rm^{-1}p$ is small; then we may expand $q_i = - p/\lambda_i$ as a power series in terms of $Rm^{-1}p$,

$$(4.1) \qquad q_i = - \frac{p}{\Lambda_i} [1 + \frac{\alpha_i}{\Lambda_i} \varepsilon + O(\varepsilon^2)] ,$$

where ε stands for $Rm^{-1}p$ and only the zeroth and the first order terms in ε are written; Λ_1 and Λ_2 are the fast and the slow speed given by [1].

$$(4.2) \qquad \Lambda_{\substack{1\\2}} = \lambda_{\substack{1\\2}} (Rm^{-1} = \infty) = \sqrt{\frac{1 + r \pm \sqrt{(1+r)^2 - 4r_x}}{2}}$$

and α_i is defined by

$$(4.3) \qquad \alpha_{\substack{1\\2}} = -(\Lambda_{\substack{1\\2}})^{-2} (1 \pm \frac{r-1}{\sqrt{(1-r)^2 + 4r_y}}) .$$

(The subscripts 1 and 2 correspond to the + and - signs in the right member, respectively.) It should be noted that the α_i's are always negative and finite. In deriving equation (4.1) we assumed not only $\varepsilon < 1$ but also

(4.3)' $\varepsilon\alpha_i < 1$.

Using the expression (4.1) for the integral (3.7) we have the saddle point given by the equation

$$1 + \frac{dq_i}{dp}\frac{x}{t} = 0$$

namely,

$$\text{(4.4)} \qquad Rm^{-1}p_i = \frac{\Lambda_i \frac{t}{x} - 1}{\alpha_i} \quad .$$

If x/t, kept constant, is sufficiently close to Λ_i, not only $Rm^{-1}p_i$ but also $Rm^{-1}p_i\alpha_i$ can be small so that the expansion used above is justified. Therefore, one can conclude that the main disturbance of the fast and the slow mode is given by the fast and the slow speeds Λ_1 and Λ_2. This result is valid for the fast wave in any case; however, for the slow wave some care must be taken. Since there is an inequality [1]

$$\text{(4.5)} \qquad \Lambda_2 \leq \min\,(1,\ b_o),$$

we may conclude that the main disturbance of the slow mode propagates with the speed Λ_2 and decreases toward the boundary determined by the sound speed, provided that the Alfven velocity is smaller than the sound velocity, $bo < 1$; however if $bo > 1$ we must be more careful. For instance, let us suppose in this case that r_y approaches zero so that Λ_2 tends to 1, then α_2 also approaches zero since $r > 1$; consequently the assumption (4.3)' ceases to be valid. From the previous result one could expect that an adequate method would give the disturbance rapidly decreasing toward the wave front $x-t = 0$, vanishing there exactly. If r_y is zero, we can see easily that equations (2.1)-(2.5) become completely separated into the sonic and the Alfven equations and the slow wave becomes the sound wave.

5. The Effect of the Hall Current

In a plasma, the Ohmic law must be modified as follows [4]

$$(5.1)^{*} \qquad \sigma(E + \frac{\mu}{c} V \times H + \frac{1}{ne} \nabla p_e) = j + \frac{e\tau\mu}{m_e c} j \times H$$

where e is the electric charge, m_e the electron mass, τ the collision frequency, and n the density of plasma.

Assuming $p_e = p/2$, and using equation (5.1) for the linearized one dimensional propagation we have

$$(5.2) \qquad (\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})\rho - b_{yo}^2 \frac{\partial^2 H_y}{\partial x^2} = 0$$

$$(5.3) \qquad (\frac{\partial^2}{\partial t^2} \quad \frac{\partial^2}{b_o^2 \partial x^2})H_y - \frac{\partial^2 \rho}{\partial x^2} - Rm^{-1} \frac{\partial^2 H_y}{\partial x^2 \partial t} - (\zeta \tan \psi) Rm^{-1} \frac{\partial^2 H_z}{\partial x^2 \partial t} = 0$$

---

[*] We neglect the inertia of electron (the plasma oscillation).

$$(5.4)\qquad \left(\frac{\partial^2}{\partial t^2} - b_{xo}^2 \frac{\partial^2}{\partial x^2}\right)H_z - Rm^{-1}\frac{\partial^2 H_z}{\partial x^2 \partial t} + (\zeta \cot \psi)Rm^{-1}\frac{\partial^2 H_y}{\partial x^2 \partial t} = 0$$

in which all the quantities x, t, ρ, H_y and H_z denote the dimensionless ones used so far, and ψ is given by the equation

$$\tan \psi = H_{zo}/H_{yo}$$

while ζ is defined by

$$\zeta \equiv \frac{\mu e H_{xo}}{m_e c} \cdot \tau \quad .$$

If the mean free path of the electron is sufficiently smaller than the Larmor radius of the electron, then ζ is small and the effects of the Hall current represented by the last terms in equations (5.2) and (5.3) can be neglected, so that the magneto-accoustic wave and the transverse wave are decoupled. However, even if ζ is not negligibly small it can be shown that there still exists a mode of propagation with the domain of dependence given by the sound velocity.

The secular equation corresponding to equation (2.8) becomes

$$(5.5)\qquad [(r_x + \varepsilon)^2 + (\zeta\varepsilon)^2]q^6 - p^2[(r_x + \varepsilon)(2 + r + \varepsilon) + (\zeta\varepsilon)^2]q^4$$
$$+ p^4(1 + r + r_x + 2\varepsilon)q^2 - p^6 = 0.$$

In the limit $|p| \to \infty$, the above equation may be approximated by

$$(\zeta^2 + 1)\varepsilon^2(q/p)^6 - (\zeta^2 + 1)\varepsilon^2(q/p)^4 + 2\varepsilon(q/p)^2 - 1 = 0$$

hence we still have one root q which behaves as $q \sim -p$ for $\mathrm{Re}\, p \to \infty$, consequently the mode corresponding to this root has a domain of dependence associated with the sound velocity.

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